The motion of a shock-wave through a region of non-uniform density

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The method of characteristics is used to calculate numerical solutions for the one-dimensional motion of a plane shock-wave through a stationary gas which contains a region of non-uniform density. These solutions are compared with those given by the Chisnell–Whitham approach which ignores the effects on the shock-wave of the disturbances which are generated in the flow behind it, and also with the asymptotic solution given by the simple theory which regards the non-uniform region as a contact-surface discontinuity. It is concluded that the results of the simplified theories must be applied with caution.

1. Introduction

Consider the one-dimensional unsteady flow in which a plane shock-wave propagates into a stationary gas which contains a stratum from $x = 0$ to $x = 1$ in which the density changes from a value ρ_0 which is constant for $x < 0$ to another value ρ_1 which remains constant for $x > 1$. Figure 1 is a representation of the resulting flow in the distance-time (x, t) plane.

The shock-wave is represented by the line OAB and, before it meets the region of non-uniform density, it has a constant shock Mach number* M_{s_0} and the flow behind it is uniform. The shock strength changes in the non-uniform region between O and A and reflected waves are produced. As these reflected waves pass through the non-isentropic region OBC between the shock and the particle path through O, further waves (which will be called re-reflected waves) are produced which travel in the same direction as the shock and eventually overtake it. These re-reflected waves influence the rate of variation of shock strength between O and A and they also cause the shock to continue to alter in strength after it has passed through the non-uniform region at A. To the left of OC, the reflected waves form a simple wave system—unless, of course, they are compression waves, in which case another shock and further re-reflected waves are eventually formed.

Chisnell (1955) obtained an expression for the change in strength of a shock-wave at an infinitesimal density discontinuity. He then integrated this expression to obtain a relationship between shock strength and density. This approach, which had previously been applied by Moeckel (1952) to the corresponding interaction in two-dimensional steady flow, ignores the influence of the re-reflected

* The shock Mach number is defined as the ratio of the speed of the shock-wave to the speed of sound in the gas in front of it.

waves. Chisnell (1957) used a similar method to treat the interaction of a shock-wave with a non-uniformity in area, and the approach was further generalized by Whitham (1958) who applied it to a large number of non-uniform shock-wave propagation problems. The results of this approach will be referred to as Chisnell–Whitham solutions.

When the density change is discontinuous, the non-uniform region reduces to a contact surface and the solution is easily obtained (Paterson 1948). As noted by Chisnell, this solution must give correctly the ultimate strength of the transmitted and reflected waves. The results of this approach will, therefore, be referred to as the asymptotic solution.

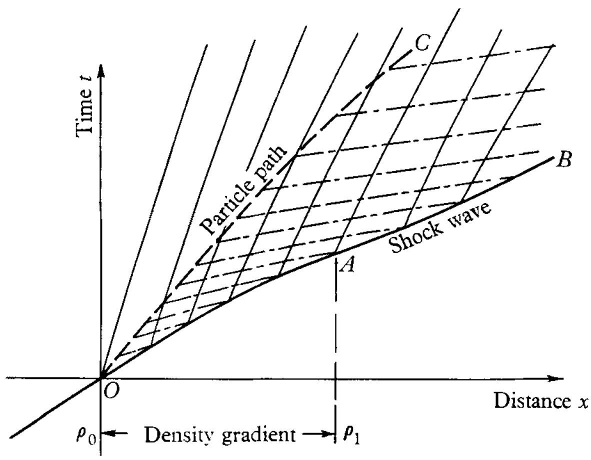

FIGURE 1. Distance–time diagram for motion of shock-wave through region of non-uniform density. ——, Reflected waves; —·—, re-reflected waves.

The Chisnell–Whitham approach gives a gradual change in shock Mach number or strength between O and A and then a constant value beyond A. In general, there is a considerable discrepancy between this value and that given by the asymptotic solution, and little is known about the rate at which the asymptotic state is reached. However, the advent of high-speed electronic computers has made it possible to obtain complete numerical solutions by the method of characteristics on a sufficiently extensive scale to enable a realistic assessment to be made of the value of the Chisnell–Whitham approach. The object of this note is to present such an assessment for this particular shock-wave interaction problem.

2. Description of method

The appropriate characteristics relationships and boundary conditions (e.g. Rudinger 1955) were set up in a form suitable for automatic computation and were programmed, using a simplified coding scheme, for the SILLIAC digital computer in the University of Sydney. The calculations were made for a perfect gas.

The programme specified the (x) mesh length along the shock-wave. This was reduced where there was a large gradient in shock strength and was made particularly small near the discontinuity in density gradient at A. Table 1 shows a typical effect of mesh size on the accuracy of the calculation. The shock Mach number quoted is that at A ($x = 1$) for $M_{s_0} = 4$, with the density varying linearly from $\rho_0 = 1$ to $\rho_1 = 0{\cdot}125$ and a specific-heat ratio $\gamma = \frac{7}{5}$.

Sufficient accuracy was obtained here for an average mesh length at the shock of less than 0·05. The solution eventually oscillated about the result given by the asymptotic solution; for reasonably small mesh sizes the amplitude of these oscillations was of the order of one-tenth of 1 % of the asymptotic shock Mach number. A typical calculation involved a total of about 500 mesh points and required about 30 min. computing time. The Chisnell–Whitham and the asymptotic solutions were also programmed for automatic computation.

Average mesh length at shock	Shock Mach number at $x = 1$
0·2	2·553
0·1	2·506
0·05	2·500
0·025	2·499

TABLE 1. Influence of mesh size.

3. Results and discussion

The accuracy of the Chisnell–Whitham approach was found to depend on the shock Mach number, the specific-heat ratio of the gas, the shape of the density profile and the magnitude of the density change.

Figure 2 shows an example (for $M_{s_0} = 1{\cdot}1$, $\rho_0 = 1$, $\rho_1 = 0{\cdot}125$ and $\gamma = \frac{7}{5}$) in which the Chisnell–Whitham solution provides an excellent description of the flow. The shock Mach number given by the characteristics solution is indistinguishable from that given by the Chisnell–Whitham solution up to the end of the non-uniform region and it then tends very slowly to the asymptotic solution —at $x = 10$ it has changed by only one-tenth of the difference. On the other hand, figure 3 (for $M_{s_0} = 4$, $\rho_0 = 1$, $\rho_1 = 32$ and $\gamma = \frac{7}{5}$) is an example in which the Chisnell–Whitham solution gives little more than the initial slope of the curve of shock Mach number against distance. At $x = 1$, the characteristics solution gives $M_s = 7{\cdot}963$ compared with $M_s = 9{\cdot}985$ by the Chisnell–Whitham method and $M_s = 7{\cdot}582$ by the asymptotic solution. If the pressure ratio across the shock were used as the measure of shock strength, the corresponding figures would be 74·98, 116·2 and 68·07. The asymptotic shock Mach number is, in fact, effectively reached by $x = 1{\cdot}5$, or in a distance beyond the end of the non-uniform layer which is much less than the thickness of the layer. This asymptotic shock strength is calculated on the assumption that the reflected waves do not form a shock-wave. However, as noted in the introduction, this is not always true but, even in this extreme example, the asymptotic shock strength would be altered by only 1·5 % to $M_s = 7{\cdot}492$. This adjustment would take place over a distance which is very large in comparison with the thickness of the non-uniform layer.

Part of the contrast between the performances of the Chisnell–Whitham approach in these two examples can be attributed to the fact that in the first the density decreases whilst in the second it increases. (Unless otherwise stated, the density variation with distance in all these examples is linear.) This effect is illustrated in figure 4 which is for $M_{s_0} = 4$, $\rho_0 = 1$, $\rho_1 = 8$ or $0{\cdot}125$, and

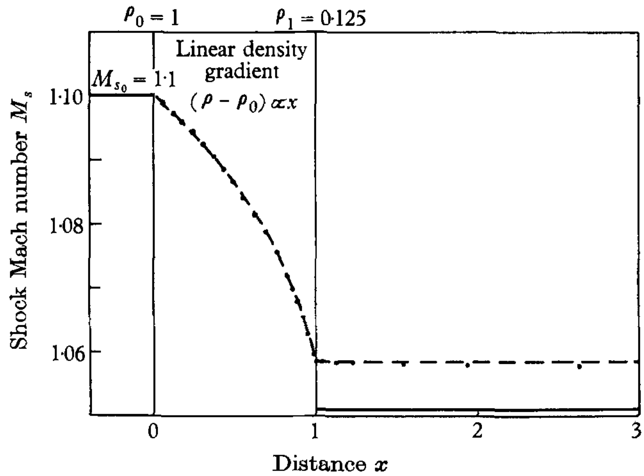

FIGURE 2. Example of good performance by Chisnell–Whitham solution. $\gamma = \frac{7}{5}$.
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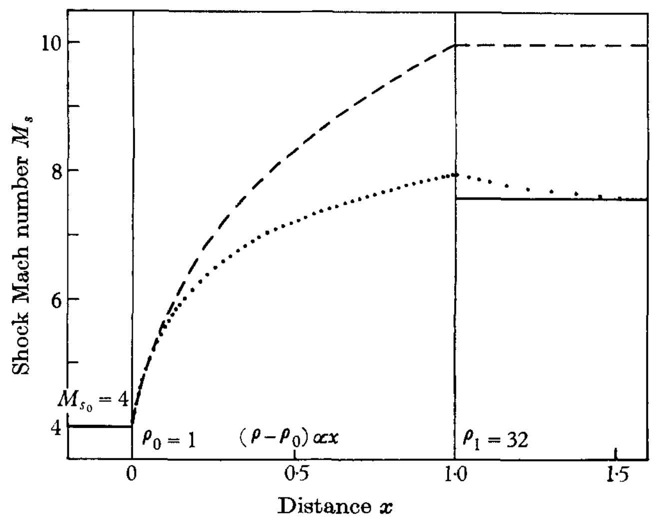

FIGURE 3. Example of poor performance by Chisnell–Whitham solution. $\gamma = \frac{7}{5}$.
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$\gamma = \frac{7}{5}$. For the positive density gradient, the shock Mach number at A $(x = 1)$ has made up 73% of the difference between the Chisnell–Whitham solution and the asymptotic solution, and the latter is effectively reached at $x = 1{\cdot}75$. However, when the density decreases the corresponding figures are 19 % and $x = 5{\cdot}5$. It is noticeable that for the increasing density the major part of the increase in shock strength occurs near $x = 0$ so that the re-reflected waves have a considerable influence on the shock before it reaches $x = 1$. Opposite considerations apply for the decreasing density and the re-reflected waves do not overtake the shock for a considerable distance. This suggested dependence on the shape of the density profile is confirmed in figure 5 which is for $M_{s_0} = 4$, $\rho_0 = 1$, $\rho_1 = 8$, $\gamma = \frac{7}{5}$

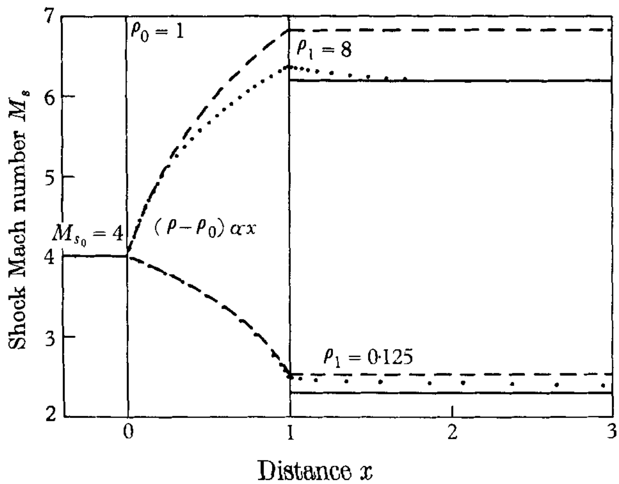

FIGURE 4. Influence of density gradient. $\gamma = \frac{7}{5}$.
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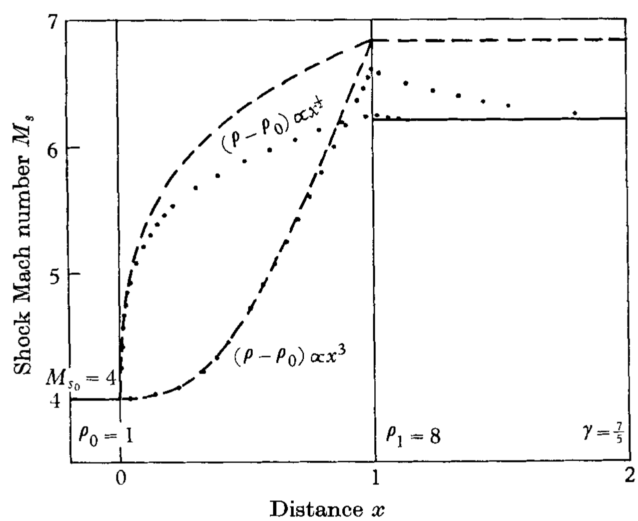

FIGURE 5. Influence of density profile. $\gamma = \frac{7}{5}$.
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and a density proportional to $x^{\frac{1}{3}}$ or x^3. The performance of the Chisnell–Whitham solution is far better when the major changes occur near the end of the layer.

Apart from this dependence on the density profile, the rate at which the asymptotic solution is approached for a given specific-heat ratio is largely a function of the shock Mach number. This is illustrated in figure 6 which shows M_s/M_{s_0} as a function of distance for $\rho_0 = 1$, $\rho_1 = 8$, $\gamma = \frac{7}{5}$ and $M_{s_0} = 1{\cdot}1$, 2 and 16.

Note that, for the increasing density, the characteristics solution is well below the Chisnell–Whitham solution even for an initial shock Mach number as low as 1·1.

Figure 7 (for $M_{s_0} = 4$, $\rho_0 = 1$, $\rho_1 = 8$ and $\gamma = \frac{5}{3}$ or $\frac{19}{17}$) illustrates the effect of specific-heat ratio. For the lower value of specific-heat ratio, the asymptotic shock Mach number is approached far more quickly than for the higher value.

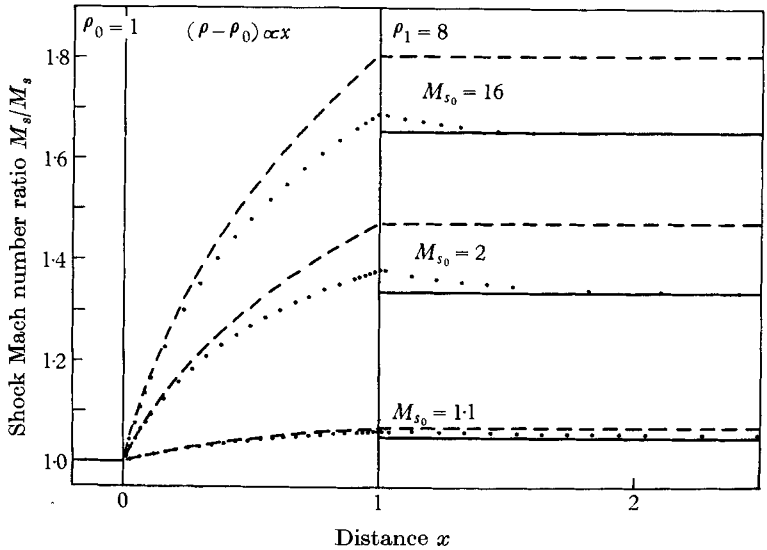

FIGURE 6. Influence of initial shock Mach number. $\gamma = \frac{7}{5}$.
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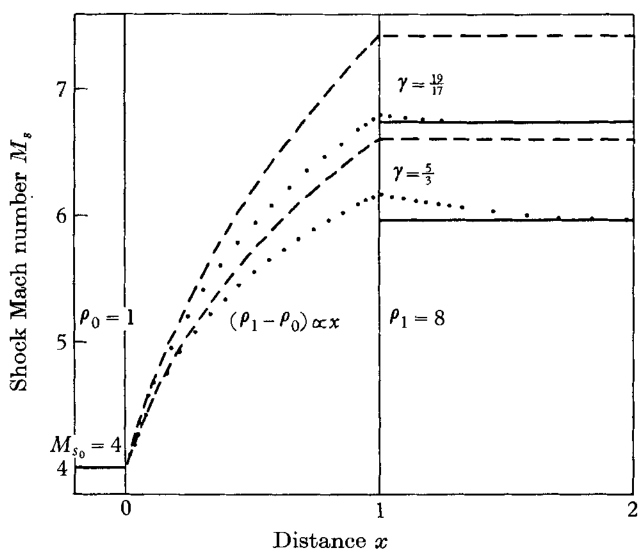

FIGURE 7. Influence of specific-heat ratio.
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The magnitude of the over-all change in density does not have any profound effect on the comparisons between the solutions. A large over-all change merely serves to magnify the differences and vice versa.

It is clear from these results that, given a combination of a weak shock-wave, a favourable density profile and a gas with a high ratio of specific heats, the Chisnell–Whitham approach can furnish an excellent quantitative description of the flow. However, for strong shock-waves in gases with low ratios of specific heats, the results given by the simple contact surface discontinuity theory can be more meaningful. If a detailed solution is required in a general case, there does not seem to be any real substitute for a full numerical analysis.

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